Sections 22.2 to 22.5

The *electric flux* Φ_E through a surface is related to electric field lines. Although we can draw as many electric field lines as we want to represent the electric field, electric flux has a definite value.

Special case 1. *If* $\vec{E}$ is uniform (that is, constant in magnitude and direction) and a flat area A is perpendicular to $\vec{E}$ as shown in Fig. 22.6a, then $\Phi_E = EA$ (in $\frac{\text{N}\cdot\text{m}^2}{\text{C}}$).

Special case 2. *If* $\vec{E}$ is uniform over a flat area A, with an angle ϕ shown in Fig. 22.6b, then $\Phi_E = EA_\perp$, where $A_\perp = A\cos\phi$. That is, $\boxed{\Phi_E = EA\cos\phi}$. Using $E\cos\phi = E_\perp$, $\boxed{\Phi_E = E_\perp A}$. Also, using $EA\cos\phi = \vec{E}\cdot\vec{A}$, $\boxed{\Phi_E = \vec{E}\cdot\vec{A}}$. These are Eqs. (22.1), (22.2), and (22.3).

That area vector $\vec{A}$ has a magnitude A (in m^2) and a direction normal (that is, perpendicular) to the surface. **For *closed* surfaces, $\vec{A}$ and $d\vec{A}$ are *always outwardly normal* to the surface** (SKILL 1).

$E_\perp$ is the component of $\vec{E}$ perpendicular to the surface (and parallel to $\vec{A}$ or $d\vec{A}$).

ϕ is the angle between the directions of the vectors $\vec{E}$ and $\vec{A}$ or $d\vec{A}$.

From $\Phi_E = EA\cos\phi$, we see the electric flux Φ_E can be $\begin{cases} +\ (90^\circ > \phi \geq 0)\text{—electric field out} \\ \quad 0\ (\phi = 90^\circ) \\ -\ (180^\circ \geq \phi > 90^\circ)\text{—electric field in} \end{cases}$ (SKILL 2) .

In general, E and/or ϕ may not be constant over the area, so $d\Phi_E = E\cos\phi\, dA$ gives Eq. (22.5), which is the mathematical *definition* of **electric flux** (TERM 1): $\boxed{\Phi_E \equiv \int E\cos\phi\, dA = \int E_\perp\, dA = \int \vec{E}\cdot d\vec{A}}$.

Cover up the solutions and carefully work Examples 22.1, 22.2, and 22.3.

In Section 22.3, our text uses $E = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{r^2}$ to find the non-integral form of **Gauss's law** (TERM 2) (in vacuum ≈ air): $\boxed{\Phi_E = \frac{Q_{\text{encl}}}{\varepsilon_0}}$, as well as the three integral forms of Gauss's law:

$\boxed{\oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0} \textbf{ and } \oint E\cos\phi\, dA = \frac{Q_{\text{encl}}}{\varepsilon_0} \textbf{ and } \oint E_\perp\, dA = \frac{Q_{\text{encl}}}{\varepsilon_0}}$. (All forms include "$= \frac{Q_{\text{encl}}}{\varepsilon_0}$ ".)

Recall that $\varepsilon_0 = 8.854 \times 10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}$.

The symbol $\oint$ means to integrate over a *closed* surface, so the electric flux Φ_E in Gauss's law is the net (that is, total) electric flux through that closed surface, and Q_{encl} is the net (that is, total) charge *enclosed* by that surface.

A **Gaussian surface** is a closed mathematical surface (TERM 3). It does not have to coincide with any material surface.

From $\Phi_E = \dfrac{Q_{\text{encl}}}{\varepsilon_0}$, we see that $Q_{\text{encl}} = \begin{cases} + \text{ gives net electric flux out} \\ 0 \text{ gives zero net electric flux} \\ - \text{ gives net electric flux in} \end{cases}$ (SKILL 2) .

Cover up the solution and carefully work Example 22.4.

$\vec{F}_0 = q_0\vec{E}$ tells us that $\boxed{E = 0 \text{ (in the material of a conductor with free charges at rest overall)}}$.

Section 22.4 explains how this $E = 0$ condition tells us that all *excess* charge is found on the surface of a *solid* conductor that has its free charges at rest overall (that is, under electrostatic conditions).

For uniformly distributed charges: the |charge| per volume is ρ (rho), $\boxed{\rho \equiv \dfrac{|q|}{V}}$ (in C/m^3) , the |charge| per area is σ (sigma), $\boxed{\sigma \equiv \dfrac{|q|}{A}}$ (in C/m^2) , and the |charge| per length is λ (lambda), $\boxed{\lambda \equiv \dfrac{|q|}{L}}$ (in C/m) , where $|q|$ is the absolute value of the charge (in C) spread uniformly over the volume V (in m^3), area A (in m^2), and/or length L (in m).

Some Applications of Gauss's law:

1. Finding electric fields from given *symmetric* charge distributions:

a) Spherical symmetry: Given a sphere of uniform positive charge density ρ and radius R.

1) Use an integral form of Gauss's law, for example, $\oint E \cos\phi \, dA = \dfrac{Q_{\text{encl}}}{\varepsilon_0}$.

2) Choose a symmetric Gaussian surface: in this case, the surface of a concentric (same center) sphere of radius r.

3) $\vec{E}$ is away from the enclosed positive charge and so is radially outward while $d\vec{A}$ is *always* outwardly normal and so is also radially outward. Therefore, the two vectors $\vec{E}$ and $d\vec{A}$ are parallel , so $\phi = 0$ and $\cos\phi = \cos 0 = 1$. Thus, $\oint E \cos\phi \, dA = \oint E\,(1)dA$.

4) E is constant by symmetry, so we can take it out of the integral: $\oint E\,(1)dA = E \oint dA$.

5) Then $\oint dA = A_{\text{spherical surface}} = 4\pi r^2$ so that the left side of Gauss's law ($\oint E \cos\phi \, dA$) equals $E4\pi r^2$. Therefore, $E4\pi r^2 = \dfrac{Q_{\text{encl}}}{\varepsilon_0}$ solves to $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_{\text{encl}}}{r^2}$ (telling us that outside of a spherically symmetric charge distribution, the electric field looks like $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{|q|}{r^2}$ —as if all the charge were concentrated at the sphere's center).

6) For $r \le R$ (inside the sphere of charge), there is charge throughout the *entire* volume of the Gaussian surface (of radius r). Thus, $Q_{\text{encl}} = \rho V_{\text{encl}} = \rho \frac{4}{3}\pi r^3$, so $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_{\text{encl}}}{r^2}$ becomes $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\rho\frac{4}{3}\pi r^3}{r^2}$ or $\boxed{E = \dfrac{\rho r}{3\varepsilon_0}}$ $(r \le R)$.

7) For $r \ge R$ (outside the sphere of charge), there is charge *only* from $r = 0$ to $r = R$ (the charge radius). Thus, $Q_{\text{encl}} = \rho V_{\text{encl}} = \rho \frac{4}{3}\pi R^3$, so $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_{\text{encl}}}{r^2}$ becomes $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\rho\frac{4}{3}\pi R^3}{r^2}$ or $\boxed{E = \dfrac{\rho R^3}{3\varepsilon_0 r^2}}$ $(r \ge R)$.

The magnitude of E has the *shape* of Fig. 22.22.

b) Cylindrical symmetry: Given a very long rod of charge of radius R, much larger length L, volume charge density ρ, and linear charge density λ.

1) Use an integral form of Gauss's law, for example, $\oint E \cos\phi \, dA = \dfrac{Q_{\text{encl}}}{\varepsilon_0}$.

2) Choose a symmetric Gaussian surface: in this case, the surface of a coaxial (same axis) cylinder of radius r and length l ($l << L$).

3) Break the integral into three parts: $\oint E \cos\phi \, dA = \int_{\text{left end}} E \cos\phi \, dA + \int_{\text{side}} E \cos\phi \, dA + \int_{\text{right end}} E \cos\phi \, dA$.

4) Over both ends: $\vec{E}$ is away from the enclosed + charge and so is radially outward. $d\vec{A}$ is always outwardly normal and so is parallel to the axis of the coaxial cylinder. Therefore, the two vectors $\vec{E}$ and $d\vec{A}$ are perpendicular , so $\phi = 90^\circ$ and $\cos\phi = \cos 90^\circ = 0$. Thus, $\int_{\text{left end}} E \cos\phi \, dA = 0 = \int_{\text{right end}} E \cos\phi \, dA$. (That is, there is no electric flux Φ_E through the ends.)

5) Over the side: $\vec{E}$ is away from the enclosed + charge and so is radially outward. $d\vec{A}$ is always outwardly normal and so is also radially outward. Therefore, the two vectors $\vec{E}$ and $d\vec{A}$ are parallel , so $\phi = 0$ and $\cos\phi = \cos 0 = 1$. Thus, $\int_{\text{side}} E \cos\phi \, dA = \int_{\text{side}} E(1) dA$.

6) E is constant by symmetry, so we can take it out of the integral: $\int_{\text{side}} E(1) dA = E \int_{\text{side}} dA$.

7) Then $\int_{\text{side}} dA = A_{\text{side}} = 2\pi rl$. Thus, the left side of Gauss's law ($\oint E \cos\phi \, dA$) equals $0 + E2\pi rl + 0 = E2\pi rl$.

8) For $r \geq R$ (outside the cylindrical charge distribution), $Q_{\text{encl}} = \rho\pi R^2 l$ or λl.

9) Thus, $\oint E \cos\phi \, dA = \dfrac{Q_{\text{encl}}}{\varepsilon_0}$ becomes $E2\pi rl = \dfrac{\rho\pi R^2 l \text{ or } \lambda l}{\varepsilon_0}$. Solving, $\boxed{E = \dfrac{\rho R^2}{2\varepsilon_0 r} \text{ or } \dfrac{\lambda}{2\pi\varepsilon_0 r}}$ $(r \geq R)$.

10) For $r \leq R$ (inside the cylindrical charge distribution), $Q_{\text{encl}} = \rho$ times what?, so $E =$ what?

c) Flat symmetry: Given a very large flat horizontal conductor with its free charges at rest overall. It has a uniform negative surface charge density $-\sigma$ on its bottom surface and no other excess charge.

1) Use an integral form of Gauss's law, for example, $\oint E \cos\phi \, dA = \dfrac{Q_{\text{encl}}}{\varepsilon_0}$.

2) Choose a symmetric Gaussian surface: in this case, the surface of a cylinder with its vertical axis perpendicular to the bottom surface of the conductor. The top and part of the side of the Gaussian cylinder are in the material of the conductor. The bottom of the Gaussian cylinder is below the flat conductor's bottom surface and has area A.

3) Break the integral into three parts: $\oint E \cos\phi \, dA = \int_{\text{top}} E \cos\phi \, dA + \int_{\text{side}} E \cos\phi \, dA + \int_{\text{bottom}} E \cos\phi \, dA$.

4) In the material of a conductor with free charges at rest overall, $E = 0$, so $\int_{\text{top}} E \cos\phi \, dA = \int_{\text{top}} 0 \cos\phi \, dA = 0$.

(That is, there is no electric flux Φ_E through the top.)

5) Part of the side is in the conductor where $E = 0$. The rest of the side is outside the conductor where $\vec{E}$ is toward the enclosed – charge, making $\vec{E}$ parallel to the side. $d\vec{A}$ is always outwardly normal and so is perpendicular to the side. Therefore, the two vectors $\vec{E}$ and $d\vec{A}$ are perpendicular to one another, so $\cos\phi = \cos 90^\circ = 0$ so $\int_{\text{side}} E\cos\phi\, dA = \int_{\text{side}} 0\, dA = 0$. (That is, there is no electric flux Φ_E through the side.)

6) Over the bottom of the Gaussian surface: $\vec{E}$ is toward the enclosed – charge and so is toward the bottom surface of the conductor. $d\vec{A}$ is always outwardly normal and so is away from the bottom surface of the conductor. Therefore, the two vectors $\vec{E}$ and $d\vec{A}$ are antiparallel (opposite) to one another, so $\phi = 180^\circ$ and $\cos\phi = \cos 180^\circ = -1$. Thus, $\int_{\text{bottom}} E\cos\phi\, dA = \int_{\text{bottom}} E(-1)dA$.

7) E is constant by symmetry, so we can take it out of the integral: $\int_{\text{bottom}} E(-1)dA = -E\int_{\text{bottom}} dA$.

8) Then $\int_{\text{bottom}} dA = A$. Thus, the left side of Gauss's law ($\oint E\cos\phi\, dA$) equals $0 + 0 + (-EA) = -EA$.

9) Since $Q_{\text{encl}} = -\sigma A$, $\oint E\cos\phi\, dA = \dfrac{Q_{\text{encl}}}{\varepsilon_0}$ becomes $-EA = \dfrac{-\sigma A}{\varepsilon_0}$, or $\boxed{E = \dfrac{\sigma}{\varepsilon_0}}$.

2. Finding a charge distribution from a known electric field:

Figure 22.23c shows the cross section of a conductor (with its free charges at rest overall). There is a cavity in the conductor that contains a charge q that is insulated from the conductor. (The charge q is shown as plus, but it could just as well be minus.) In the material of the conductor, $E = 0$, so Gauss's law ($\oint \vec{E}\cdot d\vec{A} = \dfrac{Q_{\text{encl}}}{\varepsilon_0}$) becomes $0 = \dfrac{Q_{\text{encl}}}{\varepsilon_0}$. Therefore, $Q_{\text{encl}} = 0$ (and $\Phi_E = 0$) for *all* Gaussian surfaces that are completely in the material of the conductor.

Since $Q_{\text{encl}} = 0$, the excess charge q_{cw} on the cavity wall, (that is, the *inner* surface of the conductor) is $\boxed{q_{\text{cw}} = -q}$ and **the excess charge is zero in the bulk of the material of the conductor** ($q_{\text{bulk}} = 0$). Therefore, any excess charge is only the conductor's surfaces—the cavity wall and the conductor's outer surface: $\boxed{q_{\text{total on conductor}} = q_{\text{cw}} + q_{\text{outer surface}}}$ (SKILL 7).

For example, suppose that the total excess charge on the conductor is –9 nC and there is +6 nC on the cavity wall. (That is, $q_{\text{total on conductor}} = -9$ nC and $q_{\text{cw}} = +6$ nC.)

First, the excess charge in the bulk of the conductor's material is zero.

Second, $q_{\text{cw}} = -q$ tells us the insulated charge in the cavity must be $q = -6$ nC.

Third, $q_{\text{total on conductor}} = q_{\text{cw}} + q_{\text{outer surface}}$ becomes -9 nC $= +6$ nC $+ q_{\text{outer surface}}$, so the charge on the outer surface must be $q_{\text{outer surface}} = -15$ nC. Now try Conceptual Example 22.11, covering up its solution.